

# SKEW $N$ -DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT. For a ring  $R$  with an automorphism  $\sigma$ , an  $n$ -additive mapping  $\Delta : R \times R \times \cdots \times R \rightarrow R$  is called a skew  $n$ -derivation with respect to  $\sigma$  if it is always a  $\sigma$ -derivation of  $R$  for each argument. Namely, it is always a  $\sigma$ -derivation of  $R$  for the argument being left once  $n - 1$  arguments are fixed by  $n - 1$  elements in  $R$ . In this short note, starting from Brešar Theorems, we prove that a skew  $n$ -derivation ( $n \geq 3$ ) on a semiprime ring  $R$  must map into the center of  $R$ .

## 1. INTRODUCTION

Let  $R$  be a ring with an automorphism  $\sigma$ . Recall that an additive mapping  $\mu : R \rightarrow R$  is called a  $\sigma$ -derivation if  $\mu(xy) = \sigma(x)\mu(y) + \mu(x)y$  holds for all  $x, y \in R$ . An  $n$ -additive mapping

$$\Delta : R \times R \times \cdots \times R \rightarrow R$$

(i.e., additive in each argument) is called a skew  $n$ -derivation with respect to  $\sigma$  if it is always a  $\sigma$ -derivation of  $R$  for the argument being left once  $n - 1$  arguments are fixed by  $n - 1$  elements in  $R$ . Namely, once  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in R$  are fixed, then for all  $x_i, y_i \in R$ , both

$$\Delta(a_1, \dots, x_i + y_i, \dots, a_n) = \Delta(a_1, \dots, x_i, \dots, a_n) + \Delta(a_1, \dots, y_i, \dots, a_n)$$

and

$$\Delta(a_1, \dots, x_i y_i, \dots, a_n) = \Delta(a_1, \dots, x_i, \dots, a_n) y_i + \sigma(x_i) \Delta(a_1, \dots, y_i, \dots, a_n)$$

always hold.

Note that the skew derivation is an ordinary derivation when  $\sigma$  is the identity map  $1_R$ . Naturally a skew  $n$ -derivation with respect to the identity map  $1_R$  is also called an  $n$ -derivation.

In order to illustrate the results in the literatures focused on this area clearly we will introduce some concepts related to skew  $n$ -derivations although the results in this note hold for arbitrary skew  $n$ -derivations on prime and semiprime rings. A skew  $n$ -derivation  $\Delta$  is called permuting or symmetric if

$$\Delta(x_1, x_2, \dots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$$

holds for all  $x_1, x_2, \dots, x_n \in R$  and  $\pi \in S_n$  the symmetric group of degree  $n$ . The function  $\delta : R \rightarrow R$  defined by  $\delta(x) = \Delta(x, x, \dots, x)$  is called the trace of  $\Delta$ . A

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skew 2-derivation with respect to the automorphism  $\sigma$  is also called a  $\sigma$ -biderivation. Naturally a 2-derivation is called a biderivation. Generalized  $n$ -derivations on rings can be defined similarly (see [10] for the definitions of generalized biderivations).

A ring  $R$  is called prime if  $aRb \neq 0$  for all  $a, b \in R \setminus \{0\}$ . A ring  $R$  is called semiprime if  $aRa \neq 0$  for all  $0 \neq a \in R$ . For a semiprime ring  $R$ , denote its extended centroid by  $C$  and its symmetric Martindale ring of quotients by  $Q_s$  (see [7] for reference). Particularly the extended centroid of a prime ring is a field. Denote the center of  $R$  by  $Z(R)$ . An automorphism  $\sigma$  of a semiprime ring  $R$  is called  $X$ -inner if there exists an invertible element  $p \in Q_s$  such that  $\sigma(x) = p x p^{-1}$  holds for all  $x \in R$ . Otherwise  $\sigma$  is called  $X$ -outer. For  $a, b \in R$ , write the commutator  $ab - ba$  of  $a$  and  $b$  by  $[a, b]$ . We will always use the commutator formulas  $[a, bc] = b[a, c] + [a, b]c$  and  $[ab, c] = a[b, c] + [a, c]b$  for  $a, b, c \in R$ .

The notion of a symmetric biderivation had been introduced by Maksa [24] in 1980. In 1989, Vukman [29] initiated the research of biderivations on prime and semiprime rings. He extended classical Posner Theorem [28] to symmetric biderivations in prime and semiprime rings. Thereafter many literatures were focused on biderivations of prime and semiprime rings (see [1, 2, 3, 4, 5, 6, 9, 10, 12, 13, 14, 16, 17, 19, 20, 25, 30, 32, 33, 34] for reference). Among these papers, the most important results are due to Brešar [8, 9, 10]. He gave the construction of biderivations on semiprime rings. In [15] Fošner studied the structure of generalized  $\alpha$ -derivations of prime and semiprime rings.

**Brešar Theorem** ([8, Theorem 4.1]) Let  $R$  be a semiprime ring, and let  $B : R \times R \rightarrow R$  be a biderivation. Then there exist an idempotent  $\varepsilon \in C$  and an element  $\mu \in C$  such that the algebra  $(1 - \varepsilon)R$  is commutative and  $\varepsilon B(x, y) = \mu \varepsilon[x, y]$  for all  $x, y \in R$ . Particularly if  $R$  is a noncommutative prime ring, then there exists  $\lambda \in C$  such that  $B(x, y) = \lambda[x, y]$  for all  $x, y \in R$ .

In [10] skew biderivations and inner generalized biderivations on prime rings were also characterized. So almost all results appearing in the literatures listed above can be implied by Brešar Theorems [8, 10].

In 2007, Jung and Park [18] considered permuting 3-derivations on prime and semiprime rings and obtained the following results:

**Theorem A (Jung and Park, [18, Theorem 2.3])** Let  $R$  be a noncommutative 3-torsion free semiprime ring and let  $I$  be a nonzero two-sided ideal of  $R$ . Suppose that there exists a permuting 3-derivation  $\Delta : R \times R \times R \rightarrow R$  such that  $\delta$  is centralizing on  $I$  ( $[\delta(x), x] \in Z(R), x \in I$ ), where  $\delta$  is the trace of  $\Delta$ . Then  $\delta$  is commuting on  $I$  ( $[\delta(x), x] = 0, x \in I$ ).

**Theorem B (Jung and Park, [18, Theorem 2.4])** Let  $R$  be a noncommutative 3!-torsion free prime ring and let  $I$  be a nonzero two-sided ideal of  $R$ . Suppose that there exists a nonzero permuting 3-derivation  $\Delta : R \times R \times R \rightarrow R$  such that  $\delta$  is centralizing on  $I$ , where  $\delta$  is the trace of  $\Delta$ . Then  $R$  is commutative.

Park [26] obtained the similar results for permuting 4-derivations on prime and semiprime rings. Furthermore in 2009, Park [27] considered permuting  $n$ -derivations on prime and semiprime rings.

In this short note, starting from Brešar Theorems ([8, Theorem 3.1 and 4.1]), we prove that an arbitrary skew  $n$ -derivation ( $n \geq 3$ ) on a semiprime ring  $R$  must map into the center of  $R$ . As a corollary, we obtain that an arbitrary skew  $n$ -derivation ( $n \geq 3$ ) on a noncommutative prime ring  $R$  must be zero. These results can reveal the reason why Theorem A, B and results in the literatures [26, 27] hold.

## 2. MAIN RESULT

This short note depends heavily on Brešar Theorems [8, Theorem 3.1 and 4.1]. In view of their proofs, we give a very mild modification of these two theorems in order to apply them better. The proof of [8, Theorem 3.1] implies its following form.

*Remark 2.1. (Brešar, [8, Theorem 3.1])* Let  $S$  be a set and  $R$  be a semiprime ring. If functions  $f$  and  $g$  of  $S$  into  $R$  satisfy that

$$f(s)g(t) = \xi g(s)xf(t) \text{ for all } s, t \in S, x \in R,$$

where  $\xi \in C$  is an invertible element, then there exist idempotents  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$  and an invertible element  $\lambda \in C$  such that  $\varepsilon_i \varepsilon_j = 0$  for  $i \neq j$ ,  $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$ , and

$$\varepsilon_1 f(s) = \lambda \varepsilon_1 g(s), \varepsilon_2 g(s) = 0, \varepsilon_3 f(s) = 0, (1 - \xi)\varepsilon_1 f(s) = 0$$

holds for all  $s \in S$ .

*Proof.* It is a small modification of the proof of [8, Theorem 3.1]. Define  $\varphi : E \rightarrow R$  by

$$\varphi\left(\varepsilon_1\left(\sum_{i=1}^n x_i f(s_i) y_i\right) + (1 - \varepsilon_1)r\right) = \xi \varepsilon_1\left(\sum_{i=1}^n x_i g(s_i) y_i\right) + (1 - \varepsilon_1)r,$$

and then add  $\xi$  in correspondent formulas of the proof of [8, Theorem 3.1]. At last  $(1 - \xi)\varepsilon_1 f(S) = 0$  can be deduced from

$$\left((1 - \xi)\varepsilon_1 f(t)\right)R\left((1 - \xi)\varepsilon_1 f(t)\right) = 0$$

for all  $t \in S$ . □

A modification of the proof for [8, Theorem 4.1] will give the following remark.

*Remark 2.2.* Let  $R$  be a semiprime ring with an automorphism  $\sigma$ , and let  $B : R \times R \rightarrow R$  be a  $\sigma$ -biderivation. Then there exist idempotents  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$  and invertible elements  $p \in Q_s$ ,  $\lambda \in C$  such that

- $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$ ,  $\varepsilon_1 \varepsilon_2 = \varepsilon_1 \varepsilon_3 = \varepsilon_2 \varepsilon_3 = 0$ ,
- $\varepsilon_1 B(x, y) = \varepsilon_1 p[x, y]$ ,  $\varepsilon_2 B(x, y) = 0$ , and  $\varepsilon_3[x, y] = 0$  for all  $x, y \in R$ .

*Proof.* By [10, Lemma 2.3] we have that for all  $x, y, z, u, v \in R$

$$(2.1) \quad B(x, y)z[u, v] = [\sigma(x), \sigma(y)]\sigma(z)B(u, v).$$

If  $\sigma$  is  $X$ -outer then for fixed  $x, y, u, v$  we deduce that  $B(x, y)z[u, v] = [\sigma(x), \sigma(y)]z_1 B(u, v)$  holds for all  $z, z_1 \in R$  by Kharchenko Theorem ([21, Theorem 2]). Moreover  $B(x, y)z[u, v] = 0$  holds for all  $x, y, z, u, v \in R$ . So by [7, Theorem 2.3.9 and Lemma 2.3.10] there exists an idempotent  $\varepsilon \in C$  such that  $\varepsilon B(x, y) = (1 - \varepsilon)[x, y] = 0$  holds for all  $x, y \in R$ . Setting  $p = 1$ ,  $\varepsilon_1 = 0$ ,  $\varepsilon_2 = \varepsilon$ ,  $\varepsilon_3 = 1 - \varepsilon$ , we get the conclusion in this case. If  $\sigma$  is  $X$ -inner then there exists an invertible element  $p \in Q_s$  such that  $\sigma(x) = p x p^{-1}$  for all  $x \in R$ . Now observing (2.1) we get that for all  $x, y, z, u, v \in R$

$$p^{-1}B(x, y)z[u, v] = [x, y]z p^{-1}B(u, v).$$

Following the proof of [8, Theorem 4.1] we complete the proof. □

*Remark 2.3.* From the proof of Remark 2.2 for different  $\sigma$ -biderivations  $B_1, \dots, B_t$ , the invertible element  $p$  is same when  $\sigma$  is  $X$ -inner. We can set  $p = 1$  when  $\sigma$  is  $X$ -outer. So  $p$  could be thought as same for different  $\sigma$ -biderivations.

Now we need some lemmas. Lemma 2.4 and 2.5 are used to prove Lemma 2.6. Lemma 2.6 is crucial in the proof of Theorem 2.7. The proofs are elementary computation.

**Lemma 2.4.** *Let  $R$  be a semiprime ring and  $a \in R$ . Then  $[a, [a, x]] = 0$  holds for all  $x \in R$  if and only if  $a^2, 2a \in Z(R)$ .*

*Proof.* We only deal with the “only if” part because the other part is obvious. For all  $x, y \in R$

$$(2.2) \quad \begin{aligned} 0 &= [a, [a, xy]] = x[a, [a, y]] + [a, x][a, y] + [a, x][a, y] + [a, [a, x]]y \\ &= 2[a, x][a, y]. \end{aligned}$$

Putting  $x = yz$  in (2.2) and applying (2.2) we have  $[2a, y]R[2a, y] = 0$  holds for all  $y \in R$ . Then  $2a \in Z(R)$  since  $R$  is semiprime. Moreover for any  $x \in R$ , we obtain

$$0 = [a, [a, x]] = a^2x + xa^2 - 2axa = a^2x + xa^2 - x(2a^2) = a^2x - xa^2.$$

So  $a^2 \in Z(R)$ .  $\square$

**Lemma 2.5.** *Let  $R$  be a semiprime ring with extended centroid  $C$  and  $a, b \in R$ . Then  $[a, [b, x]] = 0$  holds for all  $x \in R$  if and only if there exist idempotents  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$  and an invertible element  $\lambda \in C$  such that*

- $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1, \varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_3 = 0$  and
- $\varepsilon_1a - \lambda\varepsilon_1b, \varepsilon_2a, \varepsilon_3b, 2\varepsilon_1b, \varepsilon_1b^2 \in C$ .

*Proof.* The “if” part can be checked by direct computation. Now we consider the “only if” part. For any  $x, y \in R$

$$(2.3) \quad \begin{aligned} 0 &= [a, [b, xy]] = x[a, [b, y]] + [a, x][b, y] + [b, x][a, y] + [a, [b, x]]y \\ &= [a, x][b, y] + [b, x][a, y]. \end{aligned}$$

Putting  $x = xz$  in (2.3) and applying (2.3) we have that

$$(2.4) \quad [a, x]z[b, y] + [b, x]z[a, y] = 0$$

holds for all  $x, y, z \in R$ . By Remark 2.1 there exist idempotents  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$  and an invertible element  $\lambda \in C$  such that

- $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1, \varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_3 = 0,$
- $\varepsilon_1[a, x] = \lambda\varepsilon_1[b, x], \varepsilon_2[a, x] = 0$  and  $\varepsilon_3[b, x] = 0$  for all  $x \in R$ .

That is  $\varepsilon_1a - \lambda\varepsilon_1b, \varepsilon_2a, \varepsilon_3b \in C$ . Then for all  $x \in R$   $[\varepsilon_1b, [\varepsilon_1b, x]] = 0$  since  $\lambda$  is invertible. By Lemma 2.4 we obtain  $2\varepsilon_1b, \varepsilon_1b^2 \in C$ .  $\square$

**Lemma 2.6.** *Let  $R$  be a semiprime ring with extended centroid  $C$  and  $a, b \in R$ . Then  $[[a, x], [b, x]] = 0$  holds for all  $x \in R$  if and only if there exist an idempotent  $\varepsilon \in C$  and an element  $\zeta \in C$  such that  $\varepsilon a - \zeta\varepsilon b, (1 - \varepsilon)b \in C$ .*

*Proof.* The “if” part is obvious. Now we deal with the “only if” part. Firstly, we will prove  $[a, b] = 0$ . For any  $x, y \in R$ , we get  $[[a, x + y], [b, x + y]] = 0$ . Then for any  $x, y \in R$

$$(2.5) \quad [[a, x], [b, y]] + [[a, y], [b, x]] = 0.$$

Put  $x = xb$  in (2.5). Then for any  $x, y \in R$

$$[x[a, b] + [a, x]b, [b, y]] + [[a, y], [b, x]b] = 0.$$

That is for any  $x, y \in R$

$$x[a, b], [b, y]] + [x, [b, y]][a, b] + [a, x][b, [b, y]] + [[a, x], [b, y]]b + [b, x][[a, y], b] + [[a, y], [b, x]]b = 0.$$

Then by (2.5) for any  $x, y \in R$

$$(2.6) \quad x[a, b], [b, y]] + [x, [b, y]][a, b] + [a, x][b, [b, y]] + [b, x][[a, y], b] = 0.$$

Put  $y = b$  in (2.6). Then for any  $x \in R$

$$(2.7) \quad [b, x][[a, b], b] = 0.$$

Putting  $x = xy$  in (2.7) and applying (2.7) we have that

$$(2.8) \quad [b, x]y[[a, b], b] = 0$$

holds for all  $x, y \in R$ . Putting  $x = -[a, b]$  in (2.8), we get that

$$[[a, b], b]y[[a, b], b] = 0$$

holds for all  $y \in R$ . Then  $[[a, b], b] = 0$  since  $R$  is a semiprime ring. Putting  $y = a$  into (2.6) and applying  $[[a, b], b] = 0$  we obtain that

$$(2.9) \quad [x, [b, a]][a, b] = 0$$

holds for all  $x \in R$ . Putting  $x = xy$  into (2.9) and applying (2.9) we get that  $[x, [b, a]]y[a, b] = 0$  holds for all  $x, y \in R$ . Particularly  $[x, [b, a]]y[x, [b, a]] = 0$  for any  $x, y \in R$ . Then  $[a, b] \in Z(R)$  since  $R$  is semiprime. By  $[[a, ab], [b, ab]] = 0$  and  $[a, b] \in Z(R)$  we find  $-[a, b]^3 = 0$ . Then  $[a, b] = 0$  since  $[a, b] \in Z(R)$  and  $R$  is semiprime.

Review (2.6) then for any  $x, y \in R$

$$(2.10) \quad [a, x][b, [b, y]] + [b, x][[a, y], b] = 0.$$

Putting  $x = xz$  in (2.10) and applying (2.10) we obtain that

$$(2.11) \quad [a, x]z[b, [b, y]] + [b, x]z[[a, y], b] = 0$$

holds for all  $x, y, z \in R$ . Putting  $x = [b, x]$  in (2.11) and applying  $[[a, y], b] = -[a, [b, y]]$  (because of  $[a, b] = 0$ ), we have that

$$[a, [b, x]]z[b, [b, y]] = [b, [b, x]]z[a, [b, y]]$$

holds for all  $x, y, z \in R$ . Then by Brešar Theorem [8, Theorem 3.1] there exist idempotents  $\omega_1, \omega_2, \omega_3 \in C$  and an invertible element  $\xi \in C$  such that

- $\omega_1 + \omega_2 + \omega_3 = 1, \omega_1\omega_2 = \omega_1\omega_3 = \omega_2\omega_3 = 0,$
- $\omega_1[a, [b, x]] = \xi\omega_1[b, [b, x]], \omega_2[a, [b, x]] = 0$  and  $\omega_3[b, [b, x]] = 0$  for all  $x \in R$ .

Putting  $x = -[a, y]$  in (2.11) and then multiplying it by  $\omega_3$ , we get that

$$\omega_3[[a, y], b]z\omega_3[[a, y], b] = 0$$

holds for all  $y, z \in R$ . So  $\omega_3[a, [b, y]] = -\omega_3[[a, y], b] = 0$  since  $R$  is semiprime. Hence

$$[\omega_1a - \xi\omega_1b, [b, x]] = 0 \text{ and } [(1 - \omega_1)a, [b, x]] = 0$$

hold for all  $x \in R$ . Thus  $[a - \xi\omega_1b, [b, x]] = 0$  holds for all  $x \in R$ . Then by Lemma 2.5 there exist idempotents  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$  and an invertible element  $\lambda \in C$  such that

- $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1, \varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_3 = 0$  and
- $\varepsilon_1(a - \xi\omega_1b) - \lambda\varepsilon_1b = c_1, \varepsilon_2(a - \xi\omega_1b) = c_2, \varepsilon_3b, 2\varepsilon_1b, \varepsilon_1b^2 \in C$ .

Then

$$(\varepsilon_1 + \varepsilon_2)a = (\lambda\varepsilon_1 + \xi\omega_1(\varepsilon_1 + \varepsilon_2))(\varepsilon_1 + \varepsilon_2)b + c_1 + c_2.$$

Setting  $\varepsilon = \varepsilon_1 + \varepsilon_2$  and  $\zeta = \lambda\varepsilon_1 + \xi\omega_1(\varepsilon_1 + \varepsilon_2)$ , we complete the proof.  $\square$

**Theorem 2.7.** *A skew  $n$ -derivation ( $n \geq 3$ ) on a semiprime ring  $R$  must map into the center of  $R$ .*

*Proof.* Let  $\Delta$  be a skew  $n$ -derivation on  $R$  with respect to the automorphism  $\sigma$ . Then for fixed  $a_1, \dots, a_n \in R$  we obtain  $\Delta(a_1, \dots, a_n) = \Delta_1(a_1, a_2, a_3)$  where  $\Delta_1(x, y, z) = \Delta(x, y, z, a_4, \dots, a_n)$  is a skew 3-derivation with respect to  $\sigma$ . So it is sufficient to prove that every skew 3-derivation on  $R$  must map into the center of  $R$ . Let  $\Delta : R \times R \times R \rightarrow R$  be a skew 3-derivation with respect to  $\sigma$ . For fixed  $x_0, y_0, z_0 \in R$ , we proceed to prove that  $\Delta(x_0, y_0, z_0) \in Z(R)$ . Obviously

$$\Delta(x_0, y, z) = \varphi_{x_0}(y, z), \Delta(x, y_0, z) = \varphi_{y_0}(x, z) \text{ and } \Delta(x, y, z_0) = \varphi_{z_0}(x, y)$$

are all  $\sigma$ -biderivations on  $R$ . Then by Remark 2.2 for every  $t \in \{x_0, y_0, z_0\}$  there exist idempotents  $\varepsilon_t, \varepsilon'_t, \varepsilon''_t \in C$  and invertible elements  $p \in Q_s$ ,  $\lambda_t \in C$  such that

- $\varepsilon_t + \varepsilon'_t + \varepsilon''_t = 1$ ,  $\varepsilon_t \varepsilon'_t = \varepsilon_t \varepsilon''_t = \varepsilon'_t \varepsilon''_t = 0$ ,
- $\varepsilon_t \varphi_t(r, s) = \lambda_t \varepsilon_t p[r, s]$ ,  $\varepsilon'_t \varphi_t(r, s) = 0$  and  $\varepsilon''_t[r, s] = 0$  for all  $r, s \in R$ .

So for all  $z \in R$ , we obtain

$$\varepsilon_{x_0} \Delta(x_0, y_0, z) = \lambda_{x_0} \varepsilon_{x_0} p[y_0, z] \text{ and } \varepsilon_{y_0} \Delta(x_0, y_0, z) = \lambda_{y_0} \varepsilon_{y_0} p[x_0, z].$$

Then for all  $z \in R$ , we have

$$\lambda_{x_0} \varepsilon_{x_0} \varepsilon_{y_0} [y_0, z] = \varepsilon_{x_0} \varepsilon_{y_0} p^{-1} \Delta(x_0, y_0, z) = \lambda_{y_0} \varepsilon_{x_0} \varepsilon_{y_0} [x_0, z].$$

Hence for all  $z \in R$ , we get  $\lambda_{x_0} \varepsilon_{x_0} \varepsilon_{y_0} [y_0, z], [x_0, z] = 0$ . Then by Lemma 2.6 we obtain  $\lambda_{x_0} \varepsilon_{x_0} \varepsilon_{y_0} [x_0, y_0] = 0$ . Thus  $\varepsilon_{x_0} \varepsilon_{y_0} [x_0, y_0] = 0$  since  $\lambda_{x_0}$  is invertible. Then

$$\varepsilon_{x_0} \varepsilon_{y_0} \varepsilon_{z_0} \Delta(x_0, y_0, z_0) = \varepsilon_{x_0} \varepsilon_{y_0} (\lambda_{z_0} \varepsilon_{z_0} p[x_0, y_0]) = \lambda_{z_0} \varepsilon_{z_0} p(\varepsilon_{x_0} \varepsilon_{y_0}) [x_0, y_0] = 0.$$

Set

$$\begin{cases} \varepsilon_1 &= \varepsilon_{x_0} \varepsilon_{y_0} (1 - \varepsilon''_{z_0}) + \varepsilon'_{x_0} (1 - \varepsilon'_{y_0}) + \varepsilon'_{y_0}, \\ \varepsilon_2 &= \varepsilon_{x_0} (\varepsilon_{y_0} \varepsilon''_{z_0} + \varepsilon''_{y_0}) + \varepsilon''_{x_0} (1 - \varepsilon'_{y_0}). \end{cases}$$

It can be verified from direct computation that  $\varepsilon_1, \varepsilon_2 \in C$  are idempotents such that

- $\varepsilon_1 + \varepsilon_2 = 1$ ,
- $\varepsilon_1 \Delta(x_0, y_0, z_0) = 0$  and  $\varepsilon_2[x, y] = 0$  for all  $x, y \in R$ .

So for all  $w \in R$ , we have

$$[\Delta(x_0, y_0, z_0), w] = \varepsilon_1 [\Delta(x_0, y_0, z_0), w] + \varepsilon_2 [\Delta(x_0, y_0, z_0), w] = 0.$$

Then  $\Delta(x_0, y_0, z_0) \in Z(R)$  completes the proof.  $\square$

By Theorem 2.7 and [10, Theorem 3.2] we get the following result for prime rings.

**Theorem 2.8.** *A prime ring with a nonzero skew  $n$ -derivation ( $n \geq 3$ ) must be commutative.*

*Proof.* Let  $\Delta$  be a nonzero skew  $n$ -derivation ( $n \geq 3$ ) on a noncommutative prime ring  $R$  with respect to an automorphism  $\sigma$ . Then there exist  $a_3, \dots, a_n \in R$  such that  $\Delta_1(x, y) = \Delta(x, y, a_3, \dots, a_n)$  is a nonzero  $\sigma$ -biderivation on  $R$ . Then by Theorem 2.7 and [10, Theorem 3.2] there exists an invertible element  $p \in Q_s$  such that  $[p[x, y], z] = 0$  holds for all  $x, y, z \in R$ . Particularly for all  $x, y, z \in R$

$$0 = [p[x, yx], z] = [p[x, y]x, z] = p[x, y][x, z].$$

Moreover for all  $x, y, z \in R$  we have  $[x, y]R[x, z] = 0$  since  $p$  is invertible. So  $R$  is commutative since  $R$  is prime.  $\square$

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